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HEAT TRANSFER AT HIGH PÉCLET NUMBER IN REGIONS OF CLOSED STREAMLINES

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Abstract—An asymptotic solution is developed for the temperature distribution at high Péclet number in a doubly-connected, laminar, incompressible flow field consisting entirely of closed streamlines. It is shown that, with the exception of a thin thermal layer next to a non-isothermal surface, the temperature is constant along a streamline but that, in contrast to the analogous problem of vorticity transport, this bulk temperature distribution is in general non-uniform. It is also established that the asymptotic expression for the average Nusselt number $\overline{Nu}$ does not contain explicitly the Péclet number *Pe* irrespective of the thermal and hydrodynamic boundary conditions, a result which is at variance with what is commonly encountered in heat transfer to external flows where, as a rule, $\overline{Nu}$ increases monotonically with increasing *Pe*.

NOMENCLATURE

C, C_1,	constants;
$f(\varphi)$,	function defined by equation (8);
h_φ, h_ψ,	metrics of the coordinate system φ, ψ;
m,	0 or $\frac{1}{2}$; see equation (8);
n,	summation index;
Nu,	Nusselt number;
Pe,	Péclet number;
Re,	Reynolds number;
Pr,	Prandtl number;
T,	temperature;
$\vec{u}$,	velocity;
$\oint$,	line integral around a closed streamline.

Greek symbols

ϵ,	constant in Fourier expansion, equation (11);
φ,	a coordinate orthogonal to the stream function ψ;
ψ,	stream function;
Γ,	absolute value of the circulation along a closed streamline;
ω,	defined in equation (13);
$\vec{\omega}$,	vorticity.

Subscripts and superscripts

1,	refers to inner boundary;
2,	refers to outer boundary;
B,	refers to bulk;
φ, ψ,	refers, respectively, to the coordinates φ and ψ;
$\bar{\ }$,	average value;
$\hat{\ }$,	stretched variable.

INTRODUCTION

THIS work deals with the problem of heat transfer in two-dimensional closed streamlines, incompressible, laminar flows in the limit of high Péclet number *Pe*. It represents, therefore, a logical extension of an earlier analysis by Batchelor [1] who considered the analogous case of vorticity transfer within a domain containing entirely closed streamlines and showed, by simply integrating the Navier–Stokes equations, that the line integral $\oint(\nabla \times \vec{\omega}) \cdot d\vec{s}$ around any closed streamline must vanish identically if steady motion is to be maintained. In addition, by applying this kinematic condition to two-dimensional high-Reynolds number flows consisting of an inviscid region plus the usual thin shear layers along the

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boundaries, Batchelor was able to establish the important result that, for such flows, the vorticity is no longer arbitrary along each streamline, but must in fact distribute itself uniformly everywhere throughout this inviscid core. The presence of such a constant vorticity core has already been substantiated by Wood's [2] analytical solution to the problem of steady circulation in the region between two slightly non-coaxial rotating cylinders, as well as by Burggraf's [3] numerical results for the flow inside a fluid-filled square cavity having one moving wall.

In many respects, the corresponding thermal problem is very much akin to that of vorticity transfer, in that, for two-dimensional flows, the equations governing the transport of both vorticity and heat are of course entirely analogous. Hence, it is easy to establish [1, 3] that, in a simply connected closed streamline domain, the temperature distribution at high Pe will consist similarly, of an isothermal core surrounded by thin thermal layers along the boundaries. This analogy, however, cannot be carried over in general to the case involving a doubly connected domain, since here the existence of an isothermal core would exclude the possibility of heat being transferred from one boundary to the other. In fact, this example illustrates the fundamental difference between the boundary conditions for temperature and those for vorticity transport, in that, whereas the former can be set *a priori* irrespective of the flow, the latter are determined by the fluid motion itself. Consequently, although the net heat flux across a given closed boundary can be specified arbitrarily, that of vorticity must necessarily be zero in order to maintain steady state. We can see clearly then that the temperature–vorticity analogy can be drawn only when there is no net overall heat transfer across a closed boundary, which, of course, includes the case of a simply connected domain.

In what follows, we shall develop an expression for the asymptotic temperature distribution, with $Pe \to \infty$, in a doubly-connected flow field in which, as sketched in Fig. 1, all the streamlines enclose the inner surface. Various combinations of thermal and hydrodynamic boundary conditions will be considered. Our objective will be not only to show the interesting contrast mentioned above between two analogous heat-transfer and vorticity-transfer problems, (this will become apparent when Wood's [2] vorticity distribution in a similar system is compared with that of the present temperature field), but also to establish the asymptotic dependence of the average Nusselt number $\overline{Nu}$ on Pe since, in most heat-transfer problems, the quantity of primary interest is the value of $\overline{Nu}$ rather than the complete temperature profile.

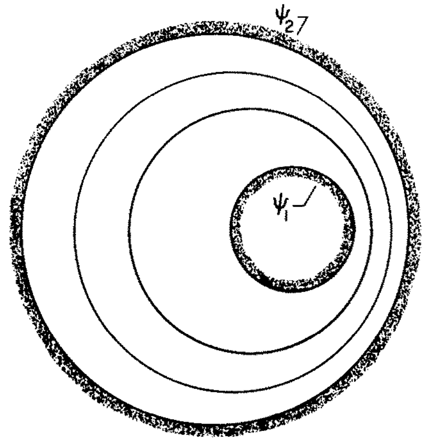

FIG. 1. A doubly-connected flow field.

ANALYSIS

We begin with the dimensionless form of the energy equation, excluding viscous dissipation,

$$\operatorname{div}(\vec{u}T) = \frac{1}{Pe}\nabla^2 T, \qquad (1)$$

and introduce the coordinate system (ψ, φ) where the constant φ curves are orthogonal to the streamlines (ψ = constant). Now, by definition, $u_\psi = 0$ and $h_\psi u_\varphi = \pm 1$, so that equation (1) reduces simply to

$$\frac{1}{Pe}\left\{\frac{\partial}{\partial\varphi}\left(\frac{h_\psi}{h_\varphi}\frac{\partial T}{\partial\varphi}\right)+\frac{\partial}{\partial\psi}\left(\frac{h_\varphi}{h_\psi}\frac{\partial T}{\partial\psi}\right)\right\} = \pm\frac{\partial T}{\partial\varphi}, \tag{2}$$

where h_φ and h_ψ denote the metrics of the transformation.

Upon examining equation (2) it becomes evident that, except perhaps near the boundaries where the gradient $\partial/\partial\psi$ may be large, the temperature up to $O(1/Pe)$, will be a function of ψ alone. This result can be interpreted to mean, physically, that at high Pe convection due to fluid circulation is much more effective than conduction normal to the stream, so that any temperature variations on the boundaries are quickly evened out a short distance away. In particular, for the simple case in which constant temperatures, T_1 and T_2, are prescribed on, respectively, the enclosing surfaces ψ_1 and ψ_2, the function

$$T = T(\psi) + O\left(\frac{1}{Pe}\right) \tag{3}$$

represents a uniformly valid solution throughout the flow domain. Moreover, it can be easily shown by integrating equation (2) around a closed streamline and making use of equation (3), that

$$T = T_1 + (T_2 - T_1)\frac{\displaystyle\int_{\psi_1}^{\psi}\frac{d\psi}{\Gamma(\psi)}}{\displaystyle\int_{\psi_1}^{\psi_2}\frac{d\psi}{\Gamma(\psi)}} \tag{4}$$

where, although $\Gamma(\psi) \equiv \oint(h_\varphi/h_\psi)\,d\varphi$, the absolute value of the circulation along the closed streamline ψ, vanishes on a stationary surface (say ψ_0) as

$$\Gamma(\psi) \to C|\psi - \psi_0|^{\frac{1}{2}} + \text{high order terms}, \tag{5}$$

the inverse of $\Gamma(\psi)$ is still integrable. Hence, this solution applies even if there is a stationary surface within the flow field, as would be the case when the two boundary surfaces move in an opposite sense.

It is easy to see now from equation (4) that the average Nusselt number corresponding to this asymptotic solution is

$$\overline{Nu} \equiv \left\{\int_{\psi_1}^{\psi_2}\frac{d\psi}{\Gamma(\psi)}\right\}^{-1} \tag{6}$$

where, as is customary, we have set $\overline{Nu}/l =$ total heat flux$/k(T_1 - T_2)$ with l and k being, respectively, the characteristic length of the system and the thermal conductivity of the fluid. Obviously, the above expression for $\overline{Nu}$ does not contain the Péclet number explicitly in contrast to the corresponding solution for external flows where, as a rule, $\overline{Nu}$ is a monotonically increasing function of Pe. On the other hand, the dependence of $\overline{Nu}$ on the Reynolds number Re can best be described by considering the following three cases:

1. $Re \to 0$. Here, the creeping flow equations apply whose solution of course does not involve the Reynolds number. Hence $\overline{Nu}$ will be independent of both Pe and Re.
2. Re moderate. Here $\overline{Nu}$ will depend on the value of Re since the latter will affect the streamline structure throughout the flow field and therefore the function $\Gamma(\psi)$ in equation (6).
3. $Re \to \infty$. Here, the flow will consist of an inviscid constant vorticity core plus shear layers of thickness $O(Re)^{-\frac{1}{2}}$ along the boundaries. However, as can be seen from equations (4) and (5), a temperature drop of at most $O(Re)^{-\frac{1}{4}}$ can exist across such a shear layer so that, for heat-transfer purposes, only the temperature distribution within the core needs to be taken into account. The latter clearly depends on the inviscid velocity profile and, hence, the resulting expression for $\overline{Nu}$, equation (6), must again be independent of both Pe and Re.

For the more general case in which a non-uniform temperature is prescribed on the boundary, the solution $T = T(\psi)$ cannot be a valid approximation everywhere even in the limit of infinite Pe, because, next to the boundary a thin thermal layer will have to exist containing large temperature gradients normal to a streamline. The analysis, therefore, requires that a boundary-layer solution be constructed and be matched to the bulk temperature distribution

$$T_B(\psi) = C_1 \int_{\psi_1}^{\psi} \frac{d\psi}{\Gamma(\psi)} + T_1, \qquad (7)$$

where T_1, the temperature on ψ_1, has been assumed uniform for simplicity, and C_1 is a constant to be determined by the matching conditions.

Following the usual procedure now, we simplify the energy equation inside the thermal layer by neglecting the temperature gradient in the longitudinal direction in comparison to that normal to the wall. Furthermore, if we restrict ourselves to the two general cases: (a) Reynolds number, Re, moderate or small but $Pe \to \infty$ (implying a very large Prandtl number Pr); (b) both $Re \gg 1$ and $Pe \gg 1$, but either $Pr \gg 1$ or $Pr \ll 1$, we can apply the usual boundary-layer arguments [4] and replace the functional coefficient h_φ/h_ψ by the simple expression

$$\frac{h_\varphi}{h_\psi} \to f(\varphi)(\psi_2 - \psi)^m. \qquad (8)$$

For case (a), $f(\varphi)$ is $O(1)$ and $m = 0$ or $\frac{1}{2}$ depending, respectively, on whether the surface ψ_2 is moving or fixed. The same also holds for the first subcase ($Pr \gg 1$) of (b) except that now $f(\varphi)$ is $O(Re)^{\frac{1}{2}}$, whereas, for second subcase of (b) where the thermal layer, although thin, is much thicker than the viscous layer, $m = 0$ and $f(\varphi)$ is $O(1)$. Admittedly, case (b) does not include the important subcase, $Re \gg 1$ and Pr of $O(1)$, which is somewhat more involved mathematically, but since, as will be seen below, the qualitative features of the $Pe \to \infty$ asymptotic solution remain unaffected by the exact form of the temperature profile inside the thermal layer, this limitation is not particularly significant.

Making use then of equation (8) and stretching the coordinates according to

$$\hat{\psi} \equiv Pe^{\frac{1}{2-m}}(\psi_2 - \psi); \qquad \hat{\varphi} \equiv \pm \int_0^{\varphi} f(\varphi)\, d\varphi \qquad (9)$$

we reduce the problem to solving

$$\frac{\partial}{\partial \hat{\psi}}\left(\hat{\psi}^m \frac{\partial T}{\partial \hat{\psi}}\right) = \frac{\partial T}{\partial \hat{\varphi}} \qquad (10)$$

with boundary conditions, in addition to the matching requirement,

$$T(0, \hat{\varphi}) = T_2(\varphi) = \sum_{n=0}^{\infty} A_n \exp\left[i(n\omega\hat{\varphi} - \epsilon_n)\right] \qquad (11)$$

$$T(\hat{\psi}, \hat{\varphi}) = T\left(\hat{\psi}, \hat{\varphi} \pm \frac{2\pi}{\omega}\right) \qquad (12)$$

where

$$\omega = 2\pi / \oint f(\varphi)\, d\varphi. \qquad (13)$$

For purposes of illustration, suppose that the Reynolds number Re is small or moderate so that $Pe \to \infty$ implies $Pr \to \infty$ with Re fixed, and that the boundary ψ_2 is non-stationary ($m = 0$). Then the solution, easily obtained by separation of variables, becomes

$$T(\hat{\psi}, \hat{\varphi}) = \sum_{n=0}^{\infty} A_n \exp\left[-\psi\sqrt{(n\omega/2)}\right] \times \cos\left\{n\omega\varphi - \psi\sqrt{(n\omega/2)} - \epsilon_n\right\} + O(Pe^{-\frac{1}{2}}) \qquad (14)$$

subject to the matching requirements,

$$\lim_{\hat{\psi}\to\infty} T(\hat{\psi}, \hat{\varphi}) = \lim_{\psi\to\psi_2} T_B(\psi) \qquad (15)$$

$$\lim_{\hat{\psi}\to\infty} Pe^{\frac{1}{2}} \frac{\partial T}{\partial \hat{\psi}} = \lim_{\psi\to\psi_2} \frac{dT_B}{d\psi}. \qquad (16)$$

However, since the expression for $T(\hat{\psi}, \hat{\varphi})$ does not involve any arbitrary constants, it is clear that, in order to satisfy equation (15) the coefficient C_1 in equation (7) must be set equal to

$$\frac{A_0 \cos \epsilon_0 - T_1}{\int_{\psi_1}^{\psi_2} \frac{d\psi}{\Gamma(\psi)}},$$

which, in turn, means that the temperature on the edge of the thermal boundary layer must be equal to the average of the specified surface temperature $T_2(\varphi)$. On the other hand, since the heat flux in the boundary layer is $O(Pe^{\frac{1}{2}})$ whereas that in the bulk is $O(1)$, the matching condition equation (16) demands that $\partial T(\hat{\psi}, \hat{\varphi})/\partial \hat{\psi}$ vanish as $\hat{\psi} \to \infty$, a requirement which is already met. Of course, since the total heat flux across any streamline must be everywhere the same, it is easy to see that the $O(Pe^{-\frac{1}{2}})$ term in equation (14) must contain a term $C_1 \hat{\psi}$ which will match exactly with the function T_B of equation (7) as $\psi \to \psi_2$.

It is seen then that although the local heat-transfer rate is proportional to $Pe^{\frac{1}{2}}$, as in the case for external flows, the average Nusselt number is again independent of Pe and is, in fact, identical to that of the constant boundary temperature solution, equation (6), if the mean of $T_2(\varphi)$ is taken to be T_2. In other words we see that, if the Péclet number is sufficiently large, a further increase in Pe does not enhance the overall rate of heat transfer from one surface to the other, but simply induces a higher influx of heat along some segments of the boundary with a corresponding release of the same amount along the remaining portions of the same boundary.

Similarly, if $T_2(\varphi)$ is given on a stationary surface (i.e. $m = \frac{1}{2}$), we obtain that

$$T(\hat{\psi}, \hat{\varphi}) = A_0 \cos \epsilon_0 + \sum_{n=1}^{\infty} \frac{4\Gamma(\frac{4}{3})}{\pi \sqrt{3}} \times [\tfrac{2}{3} \sqrt{(i\omega n)}]^{\frac{1}{3}} A_n \hat{\psi}^{\frac{1}{2}} K_{\frac{1}{3}} [\tfrac{4}{3} \hat{\psi}^{\frac{3}{4}} \sqrt{(in\omega)}] \times \exp [i(n\omega\hat{\varphi} - \epsilon_n)] + O(Pe^{-\frac{1}{3}}) \qquad (17)$$

where $K_{\frac{1}{3}}$ is the modified Bessel function of the second kind. Once again, although the local heat-transfer rate $(1/h_\psi)\, \partial T/\partial \psi |_{\psi=\psi_2}$ is $O(Pe^{\frac{1}{3}})$, the overall flux from ψ_2 to ψ_1 is not a function of Pe, and, in fact, the main difference between this and the previous case is that now the thermal boundary-layer thickness is only $O(Pe^{-\frac{1}{3}})$ rather than $O(Pe^{-\frac{1}{2}})$.

Problems involving other thermal conditions, such as prescribed heat flux on the boundary and a different range of Re, may also be treated in the same manner and yield results qualitatively similar to those shown above. It suffices, therefore, to summarize here the main conclusions of this analysis:

1. The asymptotic expression for the average Nusselt number does not contain the Péclet number explicitly irrespective of the thermal and hydrodynamic boundary conditions. Furthermore,

$$\overline{Nu} = \left\{ \int_{\psi_1}^{\psi_2} \frac{d\psi}{\Gamma(\psi)} \right\}^{-1},$$

if the temperature is normalized by either $\Delta T = \overline{T}_2(\varphi) - \overline{T}_1(\varphi)$, that is the difference between the average surface temperatures, or, in the case of prescribed flux on the boundary (say ψ_2), by

$$\Delta T = \left[\oint \frac{h_\varphi}{h_\psi} \frac{\partial T}{\partial \psi} d\varphi \right]_{\psi=\psi_2} \int_{\psi_1}^{\psi_2} \frac{d\psi}{\Gamma(\psi)}.$$

2. A thermal boundary layer exists next to a non-isothermal wall having a thickness, for small or moderate Reynolds numbers, $O(Pe^{-\frac{1}{3}})$ and $O(Pe^{-\frac{1}{2}})$, respectively, depending on whether the boundary is stationary or in motion.
3. The temperature distribution in the bulk is given in general by

$$T_B(\psi) = \overline{T}_1(\varphi) + [\overline{T}_2(\varphi) - \overline{T}_1(\varphi)] \frac{\int_{\psi_1}^{\psi} d\psi/\Gamma(\psi)}{\int_{\psi_1}^{\psi_2} d\psi/\Gamma(\psi)}. \qquad (18)$$

Hence, in contrast to Wood's [2] analogous problem of vorticity transfer at high Reynolds numbers where the solution was shown to consist of a uniform vorticity core in addition to the boundary layers next to the two surfaces, we can clearly see here that, in the case of heat transfer at high Pe, the temperature in the core will be uniform only when $\bar{T}_2(\varphi) = \bar{T}_1(\varphi)$.

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Résumé—Une solution asymptotique est présentée pour la distribution de température à des nombres de Péclet élevés dans un champ d'écoulement incompressible laminaire et à connexion double consistant entièrement en lignes de courant fermées. On montre qu'à l'exception d'une mince couche thermique près d'une surface non-isotherme, la température est constante le long d'une ligne de courant mais qu'à la différence du problème analogue du transport de vorticité, cette distribution globale de température est en général non-uniforme. On a également démontré que l'expression asymptotique pour le nombre de Nusselt moyen $\overline{Nu}$ ne contient pas explicitement le nombre de Péclet Pe indépendamment des conditions aux limites thermiques et hydrodynamiques, résultat qui est en accord avec ce que l'on rencontre habituellement dans les écoulements extérieurs où généralement $\overline{Nu}$ croît d'une façon monotone lorsque Pe augmente.

Zusammenfassung—Für die Temperaturverteilung bei grossen Péclet-Zahlen in einem zweifach zusammenhängenden, laminaren inkompressiblen Strömungsfeld, das ausschliesslich aus geschlossenen Stromlinien besteht, wurde eine asymptotische Lösung entwickelt. Es wird gezeigt, dass mit Ausnahme einer dünnen thermischen Schicht nahe einer nichtisothermen Oberfläche, die Temperatur entlang einer Stromlinie konstant ist, dass aber im Gegensatz zum analogen Problem des Wirbeltransports, diese Temperaturverteilung im Medium im allgemeinen nicht gleichmässig ist. Der asymptotische Ausdruck für die mittlere Nusselt-Zahl $\overline{Nu}$ enthält nicht explizit die Peclet-Zahl Pe unabhängig von den thermischen und hydrodynamischen Grenzbedingungen. Dieses Ergebnis weicht ab von den üblichen Erfahrungen für den Wärmeübergang bei Aussenströmung, bei dem in der Regel $\overline{Nu}$ monoton mit Pe zunimmt.

Аннотация—Предложено асимптотическое решение тепловой задачи при больших числах Пекле для ламинарного несжимаемого течения с полностью замкнутыми линиями тока. Показано, что за исключением случая тонкого термического слоя вблизи неизотермической поверхности температура постоянна вдоль линии тока. В отличие от формально аналогичной задачи переноса завихренности это распределение температуры в объеме не является равномерным. Установлено также, что асимптотическое выражение для среднего числа Нуссельта $\overline{Nu}$ не содержит в явном виде числа Пекле, независимо от тепловых и гидродинамических граничных условий. Таким образом, обнаружено отличие от обычных условий теплообмена во внешнем потоке, где $\overline{Nu}$, как правило, монотонно возрастает с числом Пекле.